Journal of Pure and Applied Algebra 38 (1985) 135-157
North-Holland

RATIONAL DEPENDENCE AMONG HILBERT AND POINCARÉ SERIES

David J. ANICK*
Department of Mathematics, Mass. Institute of Technology, Cambridge, MA 02139, USA

Tor H. GULLIKSEN
Universitetet i Oslo, Institute of Mathematics, P.O. Box 1053 Blindern, 0316 Oslo 3, Norway

Communicated by C. Löfwall
Received 5 February 1985

Dedicated to Jan-Erik Roos on his 50-th birthday

For a certain collection of sets of formal power series, we show that a series belonging to any one set is related by a rational formula to some series in any other set. The collection includes the set of Poincaré series of loop spaces on finite CW complexes; the subset obtained when we restrict to complexes of dimension four; the set of Hilbert series of finitely presented graded algebras; the set of Poincaré series of Noetherian local rings; and the subset corresponding to those rings whose maximal ideal cubed vanishes.

1. Introduction

For a certain collection of sets of formal power series, this paper shows that a series belonging to any one set of the collection is related by a rational formula to some series in any other set. This introduction is dedicated to setting forth the relevant terminology; to explaining what we intend to prove in Sections 2 and 3; and to giving the appropriate mathematical context in which these results occur. In particular, we will show how our results may be viewed as the establishment of six 'missing links' in a certain diagram.

Since the core in each of our results asserts that certain series are 'rationally related' to each other, we must define this concept first.

Definition 1. Let $A(z)=\sum_{n=0}^{\infty} a_n z^n$ and $B(z)=\sum_{n=0}^{\infty} b_n z^n$ be non-zero formal power series with (say) integral coefficients. We say that A is *rationally related* to B and write $A\sim B$ if and only if there exist polynomails $p_i(z)$, $1\le i\le 4$, such that

$$p_1(z)p_4(z)\neq p_2(z)p_3(z)$$

* Partially supported by NSF grant # DMS-8303257.

0022-4049/85/$3.30

and

$$A(z)=\frac{p_1(z)B(z)+p_2(z)}{p_3(z)B(z)+p_4(z)}.$$

A set $\mathscr{A}$ of formal power series is *rationally dependent* upon another set $\mathscr{B}$ if and only if $\forall A\in\mathscr{A}\ \exists B\in\mathscr{B}$ such that $A\sim B$. We write $\mathscr{A}\to\mathscr{B}$ when $\mathscr{A}$ is rationally dependent upon $\mathscr{B}$.

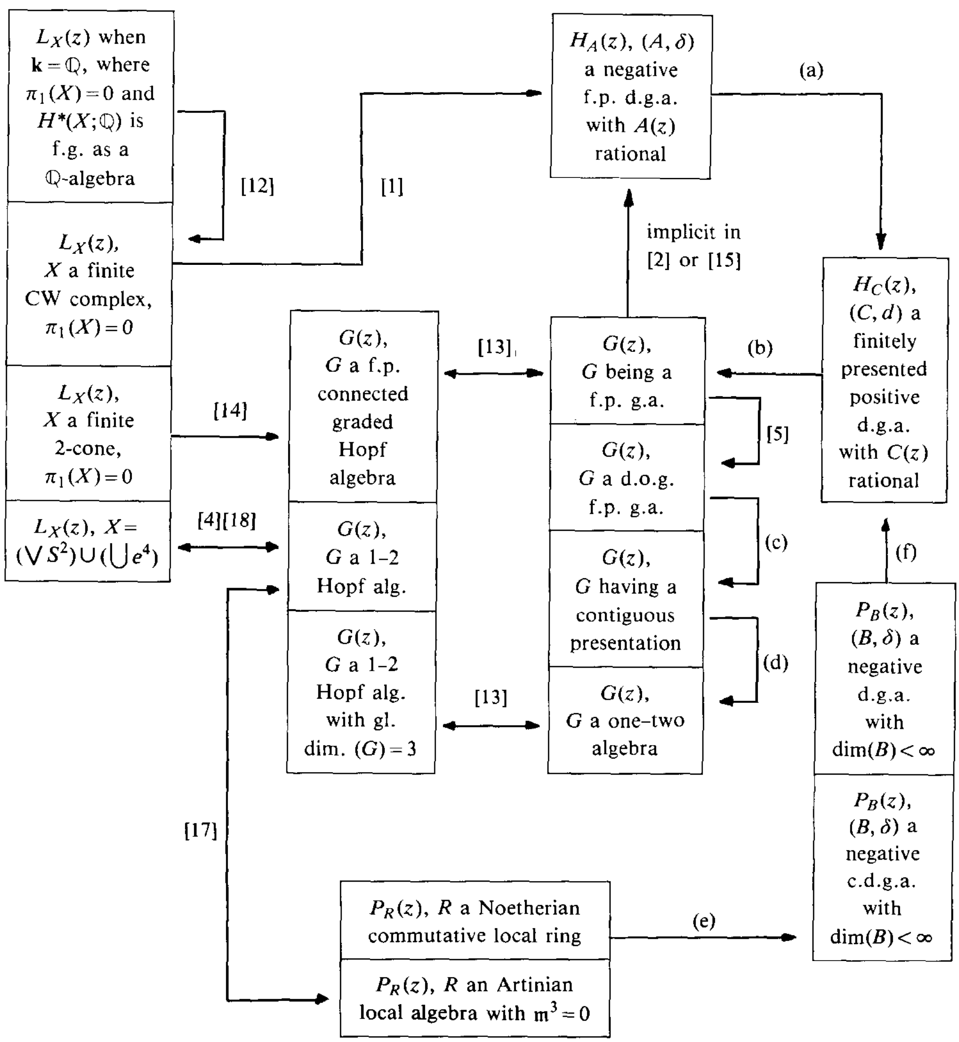

Fig. 1. A diagram showing seventeen sets of formal power series and their rational interdependencies.

Clearly $\sim$ is an equivalence relation. It is also clear that $\mathscr{A}\rightarrow\mathscr{B}$ whenever $\mathscr{A}\subset\mathscr{B}$ and that $\mathscr{A}\rightarrow\mathscr{B}\rightarrow\mathscr{C}$ implies $\mathscr{A}\rightarrow\mathscr{C}$.

The sets of formal power series which will interest us, together with their rational interconnections, are represented pictorially in Fig. 1. The best way to describe our results is to outline the meaning of each component in that picture.

Fig. 1 shows various rectangles and subrectangles with arrows between them. The areas and subareas represent sets and subsets of formal power series, and a directed path from a set $\mathscr{A}$ to a set $\mathscr{B}$ signifies that $\mathscr{A}\rightarrow\mathscr{B}$.

Divisions within the connected regions of Fig. 1 designate nested subsets, with the higher-placed labels describing more inclusive sets. For example, 'finite CW complexes' includes 'finite 2-cones', which in turn includes 'spaces of the special form $(\bigvee S^2)\cup(\bigcup e^4)$'. Directed paths representing the fact that $\mathscr{A}\rightarrow\mathscr{B}$ when $\mathscr{A}\subset\mathscr{B}$ are sometimes suppressed in the diagram, as is the path for $\mathscr{A}\rightarrow\mathscr{C}$ if $\mathscr{A}\rightarrow\mathscr{B}$ and $\mathscr{B}\rightarrow\mathscr{C}$ are both drawn.

When one includes allowance for passage from a subregion to an abutting higher subregion (representing passage from a set to a superset), one sees that one can flow from any rectangle of Fig. 1 to any other rectangle along the indicated paths. Combinatorially the diagram is called a 'strongly connected directed graph'. Paths labeled with bracketed references describe rational dependencies previously known and where the proofs can be found. Sections 2 and 3 of this paper will complete the six heretofore 'missing links' (a) through (f).

Our main result can therefore be stated as

Theorem 1. *The seventeen sets of formal power series pictured in Fig.* 1 *are all rationally dependent upon each other.* □

The set descriptions in Fig. 1 should be mostly self-explanatory once a few abbreviations are spelled out. Before proceeding we must fix a field **k**, preferably though not necessarily a prime field. The definitions of the seventeen sets will depend upon the chosen **k**, so there is actually a different version of Theorem 1 for each possible field **k**.

We will begin at the bottom of Fig. 1 and work our way upwards. To begin with, the phrase 'the set of all' is suppressed throughout for space considerations, e.g. the region labelled '$L_X(z)$, X a finite CW complex, $\pi_1(X)=0$' stands for

$$\{L_X(z)\,|\,X \text{ is a simply connected finite CW complex}\}.$$

When R is a local commutative Noetherian ring with residue field **k**, the *Poincaré series* of R is

$$P_R(z)=\sum_{n=0}^{\infty}\dim_{\mathbf{k}}(\mathrm{Tor}_n^R(\mathbf{k},\mathbf{k}))z^n.$$

In [16] G. Levin gave an important reduction theorem from which it follows that Poincaré series of Noetherian local rings are rationally related to Poincaré series of Artinian local rings.

Implicit in our Theorem 1 is the following strengthening of this result.

Theorem 2. *Given any commutative Noetherian local ring* $(S, \mathfrak{n}, \mathbf{k})$, *there is a commutative Artinian local* **k***-algebra* $(R, \mathfrak{m}, \mathbf{k})$ *with* $\mathfrak{m}^3 = 0$ *such that* $P_S(z) \sim P_R(z)$. □

In other words, all the complexity of Poincaré series of local rings is already present just among local k-algebras having $\mathfrak{m}^3 = 0$.

When X is a simply-connected locally finite CW complex, its *loop series* is the Poincaré series of its loop space ΩX:

$$L_X(z) = P_{\Omega X}(z) = \sum_{n=0}^{\infty} \dim_{\mathbf{k}}(H_n(\Omega X; \mathbf{k}))z^n.$$

The space X is a *finite* 2*-cone* if it has the homotopy type of the cofiber of a map between two finite wedges of spheres:

$$\bigvee_{j=1}^{r} S^{d_j} \xrightarrow{f} \bigvee_{i=1}^{g} S^{c_i} \to X.$$

A rather special class of spaces consists of those 2-cones for which each $d_j = 3$ and each $c_i = 2$. These spaces always have a cell decomposition of the form

$$X = \left(\bigvee_{i=1}^{g} S^2 \right) \cup \left(\bigcup_{j=1}^{r} e^4 \right). \tag{1}$$

Implicit in Theorem 1 is

Theorem 3. *Given any finite simply-connected CW complex* W (*or, if* char$(\mathbf{k}) = 0$, *any simply-connected* W *for which* $H^*(W; \mathbf{k})$ *is finitely generated as a* **k***-algebra*), *there is a space* X *of the special form* (1) *such that* $L_W(z) \sim L_X(z)$. □

As with local rings, we see that all the richness and diversity of $\{L_X\}$ is already accounted for among the loop series of spaces described by (1).

Terminology related to connected graded **k**-algebras is described in [3] and [5] and will not be repeated here. The abbreviation 'f.p.' stands for 'finitely presented'; 'g.a.' is '[connected] graded algebra'; and 'gl.dim.' means 'global dimension'. A 'd.o.g.' algebra is 'degree-one-generated'; if in addition the minimal relations are quadratic we call it a 'one–two algebra'. 'Contiguous presentation' is a technical condition which can wait until Section 3 to be defined. The *Hilbert series* of any locally finite graded vector space $M = \bigoplus_{n=0}^{\infty} M_n$ is

$$M(z) = \sum_{n=0}^{\infty} \dim(M_n)z^n.$$

A Hilbert series is *rational* if and only if it equals the power series expansion around zero of a rational function. In particular, by [3, Lemma 1.2] the Hilbert series of a f.p. free algebra is always rational.

The abbreviation 'd.g.a.' stands for '[connected associative] differential graded **k**-algebra.' A *d.g.a.* is a graded algebra A together with a **k**-linear map $d\colon A \to A$ of degree ± 1 such that $d^2 = 0$ and d is a derivation, i.e.,

$$d(xy) = d(x)y + (-1)^{|x|} x d(y) \tag{2}$$

for all homogeneous x and y in A. We call (A, d) a *positive* d.g.a. if $|d| = +1$ and (A, d) is a *negative* d.g.a. if $|d| = -1$.

(*Note.* This terminology, while reasonable, conflicts with another convention which assumes that $|d|$ is always minus one. What we call a 'positive d.g.a.' becomes under that convention a d.g.a. which is 'negative' in the sense that its non-zero part is in negative degrees.) In either case, (A, d) is a *commutative differential graded algebra* (c.d.g.a.) if for all homogeneous x and y in A we have

$$xy = (-1)^{|x| \cdot |y|} yx,$$

and $x^2 = 0$ whenever $|x|$ is odd. The *homology series* of (A, d) is

$$H_A(z) = \sum_{n=0}^{\infty} \dim_{\mathbf{k}}(H_n(A, d)) z^n,$$

and when $|d| = -1$ its *Poincaré series* is the Hilbert series of its differential graded torsion, i.e.,

$$P_A(z) = \sum_{n=0}^{\infty} \dim_{\mathbf{k}}(\operatorname{Tor}_n^{(A,d)}(\mathbf{k}, \mathbf{k})) z^n$$

(For the definition of differential graded torsion see [8] or [9]; for advice on computing it in the commutative case, see, e.g., [11, Theorem 1.6.2].) Thus a negative d.g.a. has three series associated with it: its Poincaré series $P_A(z)$, its homology series $H_A(z)$, and the ordinary Hilbert series $A(z)$ of the graded algebra A.

This completes a brief overview of the classes of series we shall consider.

One of the paths signifying rational dependence in Fig. 1 deserves further clarification. The link showing that any $G(z)$ for a one–two Hopf algebra G is rationally related to the loop series for a space of the form $(\bigvee S^2) \cup (\bigcup e^4)$ is proved in [18] only for $\mathbf{k} = \mathbb{Q}$, and it is proved implicitly in [14] or more explicitly in [4] only for prime fields **k**. Because the usual method for topologically realizing the Hilbert series of a one–two Hopf algebra involves recasting the coefficients in G's presentation as the coefficients in a sum of Whitehead products, it requires that this presentation utilize only the images in **k** of rational integers (cf. [2, Lemma 5.7]). Only when **k** is a prime field can we count on an arbitrary G to have such a presentation. Thus for non-prime fields we cannot necessarily assert that all the other series are rationally related to loop series, even though the other rational interdependencies remain intact.

2. The connection with differential graded algebras

This section is devoted to proving links (a), (b), (e) and (f) of Fig. 1. We will do this by describing four constructions, one from positive d.g.a.'s to graded algebras; the next from negative to positive d.g.a.'s; the third from Noetherian local rings $(R, \mathfrak{m}, \mathbf{k})$ to negative commutative d.g.a.'s of finite dimension over $\mathbf{k}$; and finally the well-known bar-(cobar-)construction.

Theorem 4. *Let (C, d) be any positive d.g.a. over $\mathbf{k}$. There is a graded algebra $\tilde{C}$ such that*

(A) *If C if f.p. as a graded algebra, so is $\tilde{C}$.*

(B) *If C is a one–two algebra, so is $\tilde{C}$.*

(C) *The construction is functorial. That is, whenever $\phi:(C,d)\to(C',d')$ is a homomorphism of positive d.g.a.'s, there is an induced homomorphism $\tilde{\phi}:\tilde{C}\to\tilde{C}'$.*

(D) *The Hilbert series of $\tilde{C}$ is related to the Hilbert series of the algebra C and to the homology series of (C, d) by the formula*

$$\tilde{C}(z)=(1+z)C(z)+\frac{z(1+z)C(z)^2}{(1+z)^2-zC(z)-z^2H_C(z)}. \tag{3}$$

(E) *In particular, if $C(z)$ is rational, then $\tilde{C}(z)\sim H_C(z)$.*

Corollary 1 (Link (b)). *The set of homology series of finitely presented positive d.g.a.'s with rational Hilbert series is rationally dependent upon the set of Hilbert series of finitely presented graded algebras.*

Proof. Theorem 4(A) and 4(E). □

Corollary 2. *The set of homology series of positive d.g.a.'s (C, d) for which $C(z)$ is rational and C is a one–two algebra is rationally dependent upon the set of Hilbert series of one–two algebras.*

Proof. Theorem 4(B) and 4(E). □

Proof of Theorem 4. Given (C, d), define $\tilde{C}$ by

$$\tilde{C}=(C\amalg\mathbf{k}\langle x,y\rangle)/J,$$

where $|x|=|y|=1$ and J is the two-sided ideal of $C\amalg\mathbf{k}\langle x,y\rangle$ generated by

$$\{x^2,xy,yx,y^2\}\cup\{\text{all } [x,c]-d(c)\,|\,c \text{ is homogeneous, } c\in C\}.$$

Here $[a,b]$ denotes $ab-(-1)^{|a|\cdot|b|}ba$, so $[x,c]=xc-(-1)^{|c|}cx$.

To prove parts (A) and (B), put $D(c)=[x,c]-d(c)$ and note that D is a derivation, so the relations $\{[x,c]-d(c)\}$ are all consequence of the relations $\{[x,x_i]-d(x_i)\}$ as x_i runs through a set of generators for C. If $C=\mathbf{k}\langle x_1,\dots,x_g\rangle/\langle\alpha_1,\dots,\alpha_r\rangle$ is a presentation for C, then a presentation for $\tilde{C}$ is

$$\tilde{C}=\mathbf{k}\langle x_1,\dots,x_g\rangle/\langle\alpha_1,\dots,\alpha_r;\,x^2,xy,yx,y^2;\,[x,x_1]-d(x_1),\dots,[x,x_g]-d(x_g)\rangle.$$

In particular, (A) and (B) follow.

To prove part (C), any $\phi:(C,d)\to(C',d')$ induces a canonical map $\hat{\phi}$: $C\amalg \mathbf{k}\langle x,y\rangle \to C'\amalg \mathbf{k}\langle x,y\rangle$ satisfying $\hat{\phi}(x)=x$, $\hat{\phi}(y)=y$, and $\hat{\phi}(c)=\phi(c)$ for $c\in C$. Because

$$\hat{\phi}([x,c]-d(c))=[\hat{\phi}(x),\hat{\phi}(c)]-\phi(d(c))=[x,\phi(c)]-d'(\phi(c)),$$

$\hat{\phi}$ induces a homomorphism on the quotient algebras, $\hat{\phi}:\tilde{C}\to\tilde{C}'$.

Part (E) of Theorem 4 is obviously an immediate consequence of part (D), so it remains only to compute $\tilde{C}(z)$. To facilitate this, we need two additional notations. For a graded vector space $N=\bigoplus_{n=0}^{\infty} N_n$ let sN denote its suspension, i.e., $sN=\bigoplus_{n=0}^{\infty}(sN)_n$ where $(sN)_0=0$ and $(sN)_n=N_{n-1}$ for $n>0$. For a graded vector space N having $N_0=0$, let TN denote the connected graded tensor algebra generated by N, i.e., $TN=\bigoplus_{i=0}^{\infty}(N^{\otimes i})$.

Put $B=\operatorname{im}(d)\subset C$ and let $\bar{C}$ be the quotient vector space

$$\bar{C}=C/(C_0\oplus B).$$

Consider the graded vector space

$$M=C\oplus sC\oplus(C\otimes T(s\bar{C})\otimes sC),$$

in which we write a typical element as a linear combination of elements having the special form

$$u=a\oplus sb\oplus(c_1\otimes w\otimes sc_2). \tag{4}$$

Here a, b, c_1, $c_2\in C$ and w is a product

$$w=(s\bar{w}_1)(s\bar{w}_2)\dots(s\bar{w}_q),$$

w_j belonging to C and $\bar{w}_j$ denoting the image of w_j in $\bar{C}$. In order for (4) to be homogeneous of degree n in M, we require that $n=|a|=|b|+1=|c_1|+|w|+|c_2|+1$, where $|w|=|w_1|+\dots+|w_q|+q$.

We can make M into a left $(C\amalg\mathbf{k}\langle x,y\rangle)$-module as follows. For $c\in C$ put

$$c*(u)=ca\oplus s(cb)\oplus(cc_1)\otimes w\otimes sc_2.$$

when u is given by (4). Also put

$$x*(u)=d(a)\oplus s(d(b)+(-1)^{|a|}a)\oplus d(c_1)\otimes w\otimes sc_2$$

and

$$y*(u)=0\oplus 0\oplus 1\otimes[1\otimes s(a-(-1)^{|b|}d(b))+((s\bar{c}_1)(w)\otimes sc_2)].$$

A straightforward calculation which we leave to the reader shows that M is annihilated by the ideal J. Hence M may be considered as a left module over the ring $\tilde{C}$.

In order to compute the Hilbert series $\tilde{C}(z)$ we will define a surjective homomorphism of graded vector spaces $\psi:\tilde{C}\to M$ which will turn out to be an isomorphism.

Let 1 continue to denote the unit element of the algebra C and let $e=1\oplus 0\oplus 0\in M$. It turns out that M is a cyclic $(C\amalg \mathbf{k}\langle x,y\rangle)$-module. To see this let a, b, c_1, w_1, $w_2,\dots,w_q$, $c_2\in C$ satisfy

$$|a|=|b|+1=|c_1|+|w_1|+\dots+|w_q|+|c_2|+q+1.$$

Then

$$(a+bx+c_1yw_1y\dots w_qyc_2)*(e)=a\oplus sb\oplus c_1\otimes(s\bar{w}_1)\dots(s\bar{w}_q)\otimes(sc_2).$$

Hence we have a surjection of left $\tilde{C}$-modules $\psi:\tilde{C}\to M$ defined by $\psi(1)=e$. In particular, we have

$$\dim(\tilde{C}_n)\geq\dim(M_n)\quad\text{in every degree } n. \tag{5}$$

We can obtain the reverse inequality by choosing a homogeneous basis $\hat{S}$ for $B\oplus C_0$; extending to a basis S for C; defining $\bar{S}=S-\hat{S}$, which surjects to a basis for $\bar{C}$; and considering the following subset of $C\amalg\mathbf{k}\langle x,y\rangle$:

$$\tilde{S}=\{a\mid a\in S\}\cup\{bx\mid b\in S\}$$
$$\cup\{c_1(yw_1)\dots(yw_q)yc_2\mid q\geq 0,\ w_j\in\bar{S},\ c_1\in S,\ c_2\in S\}.$$

We assert that the image of $\tilde{S}$ in $\tilde{C}=(C\amalg\mathbf{k}\langle x,y\rangle)/J$ spans $\tilde{C}$. This is equivalent to asserting that $C\amalg\mathbf{k}\langle x,y\rangle=J+\operatorname{span}(\tilde{S})$. To see this, first consider elements of $C\amalg\mathbf{k}\langle x,y\rangle$ which do involve x. Because $xc\equiv\pm cx+d(c)$ (mod J) and $xcx\equiv d(c)x\pm cx^2\equiv d(c)x$ (mod J), and similar formulas hold for xcy or ycx, we can always reduce to the case where a word involving x non-trivially has the special form bx with $b\in C$. Thus the two-sided ideal generated by x in $C\amalg\mathbf{k}\langle x,y\rangle$ is contained in $(J+\operatorname{span}(\tilde{S}))+(C\amalg\mathbf{k}\langle y\rangle)$.

It remains to show that $C\amalg\mathbf{k}\langle y\rangle)\subset J+\operatorname{span}(\tilde{S})$. A basis for $C\amalg\mathbf{k}\langle y\rangle$ is certainly

$$\{a\mid a\in S\}\cup\{c_1(yw_1)\dots(yw_q)yc_2\mid c_1,c_2,w_j\in S\}.$$

But if any $w_j\in\hat{S}$, then $w_j\in B\oplus C_0=\operatorname{im}(d)\oplus\mathbf{k}$, so either $yw_jy=y^2$ or $yw_jy=yd(c)y\equiv yxcy\pm ycxy$ (mod J) for some c. In either case $yw_jy\in J$, so $c_1yw_1\dots yw_qyc_2\in J$. The assertion that $C\amalg\mathbf{k}\langle y\rangle\subset J+\operatorname{span}(\tilde{S})$ follows. We have shown that $\tilde{S}$, which can obviously serve as a basis for M, is a spanning set for $\tilde{C}$. Deduce that

$$\dim(M_n)\geq\dim(\tilde{C}_n)\quad\text{for every } n. \tag{6}$$

Combining inequalities (5) and (6) we obtain

$$\dim(M_n)=\dim(\tilde{C}_n)\quad\text{for all } n,$$

leading to

$$\tilde{C}(z)=M(z).$$

But $M(z)$ is easy to compute! It is

$$M(z)=C(z)+zC(z)+C(z)(1-z\bar{C}(z))^{-1}zC(z). \tag{7}$$

By formula (8), whose proof follows immediately, we have

$$\begin{aligned}\bar{C}(z)&=C(z)-B(z)-1\\&=C(z)-\frac{zC(z)}{1+z}+\frac{zH_C(z)}{1+z}-1\\&=\frac{C(z)+zH_C(z)-(1+z)}{1+z},\end{aligned}$$

from which (3) follows by substitution into (7). □

Lemma 1. *Let (C, d) be a positive or negative d.g.a. and let $B=\operatorname{im}(d)$ be the graded submodule of boundaries. If $|d|=+1$, then*

$$B(z)=\left(\frac{z}{1+z}\right)(C(z)-H_C(z)) \tag{8}$$

and if $|d|=-1$, then

$$B(z)=\left(\frac{1}{1+z}\right)(C(z)-H_C(z)). \tag{9}$$

Proof. First suppose $|d|=+1$. Write d_n for $d|_{C_n}$ and note that for each n we have a short exact sequence of graded **k**-modules

$$0\to \operatorname{Ker}(d_n)\to C_n\to B_{n+1}\to 0.$$

Putting $K=\operatorname{Ker}(d)$ we get

$$K(z)=C(z)-z^{-1}B(z).$$

Because $H_C(z)=K(z)-B(z)$ we have

$$\begin{aligned}H_C(z)&=C(z)-z^{-1}B(z)-B(z)\\&=C(z)-z^{-1}(1+z)B(z),\end{aligned}$$

from which we obtain formula (8).

The proof when $|d|=-1$ is similar. □

Our approach to negative d.g.a.'s is to turn them into positive d.g.a.'s by adjoining a special element of degree two.

Theorem 5. *Let (A, δ) be a negative differential graded algebra. Then there is a positive differential graded algebra (C, d) such that*

(A) *If A is finitely presented, so is C.*

(B) *The construction is functorial, i.e., any homomorphism* $\phi:(A,\delta)\to(A',\delta')$ *induces a homomorphism* $\hat{\phi}:(C,d)\to(C',d')$.

(C) *The Hilbert series of C is* $C(z)=A(z)/(1-z^2)$. *In particular,* $C(z)$ *is rational if* $A(z)$ *is rational.*

(D) *The homology series of C satisfies*

$$H_C(z)=\frac{A(z)}{1+z}+\frac{zH_A(z)}{1-z^2}. \tag{10}$$

(E) *If* $A(z)$ *is rational, then* $H_A(z)\sim H_C(z)$.

Corollary 3 (Link (a)). *The set of homology series of finitely presented negative d.g.a.'s having rational Hilbert series is rationally dependent upon the set of homology series of finitely presented positive d.g.a.'s with rational Hilbert series.*

Proof. Theorem 5(A) and 5(C) and 5(E). □

Proof of Theorem 5. Given (A,δ), define C by $C=A\otimes\mathbf{k}[y]$, where y is a new generator of degree two. Claims 5(A) and 5(C) and the implication 5(D) ⇒ 5(E) should be clear. Define d by

$$d(uy^a)=\delta(u)y^{a+1}$$

for $u\in A$. The reader can check that $d^2=0$ and that d is a derivation, so (C,d) is indeed a positive d.g.a.

If $\phi:(A,\delta)\to(A',\delta')$ is a homomorphism of negative d.g.a.'s, i.e., if $\phi\delta=\delta'\phi$, then the algebra map

$$\hat{\phi}:(A\otimes\mathbf{k}[y],\ d)\to(A'\otimes\mathbf{k}[y],\ d')$$

given by $\hat{\phi}|_A=\phi$ and $\hat{\phi}(y)=y$ satisfies

$$\hat{\phi}d(uy^a)=\hat{\phi}(\delta(u)y^{a+1})=\phi(\delta(u))y^{a+1}=\delta'(\phi(u))y^{a+1}=d'(\hat{\phi}(uy^a)),$$

so 5(B) holds.

It remains only to calculate $H_C(z)$. But it is obvious that the $\mathbf{k}$-modules $D=\mathrm{im}(d)$ and $B=\mathrm{im}(\delta)$ are related via

$$D=\{uy^a\mid u\in B \text{ and } a\geq 1\}.$$

Consequently

$$D(z)=\frac{z^2B(z)}{(1-z^2)}.$$

By formula (9) we have

$$B(z)=(1+z)^{-1}(A(z)-H_A(z))$$

and by (8)

$$D(z)=z(1+z)^{-1}(C(z)-H_C(z)),$$

so (10) follows by solving for $H_C(z)$. □

Let us turn now to the lower right portion of Fig. 1. In relating Poincaré series of local rings to homology series of finitely presented positive d.g.a.'s we originally worked in the equi-characteristic case. It was the idea of Clas Löfwall to use the next two results to get a reduction which is valid in any characteristic. Link (e) clearly follows from the following theorem.

Theorem 6. *Let $(R, \mathfrak{m}, \mathbf{k})$ be any commutative Noetherian local ring, and let $n=\dim(\mathfrak{m}/\mathfrak{m}^2)$ be its embedding dimension. Then there is a negative commutative differential graded algebra B which is finite-dimensional over $\mathbf{k}$ such that $P_R(z)=(1+z)^n P_B(z)$.*

Proof. This result is essentially Corollary 4.6 in [10]. We include a slightly different proof pointed out to us by Löfwall.

Without loss of generality we can assume that R is complete in the $\mathfrak{m}$-adic topology. Hence R is a homomorphic image of a regular local ring $\tilde{R}$ of the same embedding dimension. Let $\tilde{K}$ be the Koszul complex of $\tilde{R}$ and let $K=\tilde{K}\otimes_{\tilde{R}}R$. Since the acyclic closure of the augmented R-algebra $R\to\mathbf{k}$ is obtained by first forming the Koszul complex K by adjoining variables of degree 1 and then forming the acyclic closure of $K\to\mathbf{k}$, we obtain

$$P_R(z)=(1+z)^n P_K(z).$$

We refer to [11] for details on acyclic closure. Let L be the acyclic closure of the augmented $\tilde{R}$-algebra $\tilde{R}\to R$. Then L is a commutative d.g. algebra which is free over $\tilde{R}$. Put $A=\mathbf{k}\otimes_{\tilde{R}} L$. The canonical augmentation maps $L\to R$ and $\tilde{K}\to\mathbf{k}$ induce quasi-isomorphisms

$$K=\tilde{K}\otimes_{\tilde{R}} R \xleftarrow{\sim} \tilde{K}\otimes_{\tilde{R}} L \xrightarrow{\sim} \mathbf{k}\otimes_{\tilde{R}} L=A$$

giving rise to an isomorphism

$$\mathrm{Tor}^{(K,d_K)}(\mathbf{k},\mathbf{k})\cong\mathrm{Tor}^{(A,d_A)}(\mathbf{k},\mathbf{k}).$$

This yields $P_K(z)=P_A(z)$. Since K is zero in degrees above n we have

$$H_i(A)\cong H_i(K)=0 \quad \text{for } i>n.$$

Thus we have an acyclic ideal I in A defined by $I_i=A_i$ for $i>n$, $I_n=\mathrm{im}(d_A)\cap A_n$, $I_i=0$ for $i<n$. This yields another quasi-isomorphism $A\xrightarrow{\sim}A/I$. Setting $B=A/I$ we have $P_B(z)=P_A(z)$. The desired formula follows and the proof is complete since B is a negative commutative d.g.a. which is of finite dimension over $\mathbf{k}$. □

Link (f) can be described quite simply as the dual of the bar construction. However, although the properties of this construction are well-known, the authors were unable to find the result we need spelled out explicitly in any one reference. For this reason we supply a proof here.

Proposition. *For any finite-dimensional negative differential graded algebra* (B,δ), *there is a finitely generated free positive d.g.a.* (C,d) *such that* $P_B(z)=H_C(z)$.

Corollary 4 (Link (f)). *The set of Poincaré series of finite-dimensional negative d.g.a.'s is rationally dependent upon the set of homology series of finitely presented positive d.g.a.'s* (C,d) *having* $C(z)$ *rational.* □

Proof of Proposition. The bar construction [7, p. 73] on any associative augmented negative d.g.a. (B,d_B) gives a differential graded coalgebra. To get it explicitly, one forms the tensor algebra $Ts\bar{B}$, $s\bar{B}$ denoting the suspension of the augmentation ideal $\bar{B}$. One then defines a differential $\hat{d}$ having $|\hat{d}|=-1$ on $B\otimes Ts\bar{B}$ by $\hat{d}=d_1+d_2$, where d_1 and d_2 are given by these formulas:

$$d_1(a\otimes sb_1\otimes\dots\otimes sb_t)=d_B(a)\otimes sb_1\otimes\dots\otimes sb_t + \sum_{i=1}^{t}\lambda(i)a\otimes sb_1\otimes\dots\otimes sd_B(b_i)\otimes\dots\otimes sb_t;$$

$$d_2(a\otimes sb_1\otimes\dots\otimes sb_t)=\lambda(1)ab_1\otimes\dots\otimes sb_t + \sum_{i=2}^{t}\lambda(i)a\otimes sb_1\otimes\dots\otimes s(b_{i-1}b_i)\otimes\dots\otimes sb_t;$$

where $\lambda(i)=(-1)^{i+|a|+|b_1|+\dots+|b_{i-1}|}$.

It is straightforward to verify the following properties:

(A) $d_1^2=d_2^2=d_1d_2+d_2d_1=0$, so $\hat{d}^2=0$.

(B) Viewing $B\cong B\otimes \mathbf{k}$ as a submodule of $B\otimes Ts\bar{B}$, $\hat{d}|_{B\otimes\mathbf{k}}=d_B$.

(C) $(B\otimes Ts\bar{B},\hat{d})$ is acyclic. If $\eta: B\to\bar{B}$ is any left inverse to the inclusion $\bar{B}\hookrightarrow B$ such that $\eta d_B=d_B\eta$ (which exists in our case because we assume any d.g.a. to be connected), then the map $a\otimes sb_1\otimes\dots\otimes sb_t\mapsto 1\otimes s\eta(a)\otimes sb_1\otimes\dots\otimes sb_t$ is a chain contraction.

(D) $(B\otimes Ts\bar{B},\hat{d})$ is a left (B,d_B)-module.

Properties (A)–(D) show that $(B\otimes Ts\bar{B},\hat{d})$ is a (B,d_B)-resolution of $\mathbf{k}$. Therefore

$$\mathrm{Tor}_*^{(B,d_B)}(\mathbf{k},\mathbf{k})\cong H_*((\mathbf{k},0)\otimes_B(B\otimes Ts\bar{B},\hat{d}))=H_*(Ts\bar{B},\tilde{d}),$$

where $\tilde{d}=\mathrm{id}_\mathbf{k}\otimes_B\hat{d}$.

The key property of $\tilde{d}$ is that it makes $Ts\bar{B}$ into a differential graded coalgebra, i.e., the coassociative diagonal map

$$\Delta: Ts\bar{B}\to Ts\bar{B}\otimes Ts\bar{B}$$

given by

$$\Delta(sb_1\otimes\dots\otimes sb_t)=\sum_{i=0}^{t}(sb_1\otimes\dots\otimes sb_i)\otimes(sb_{i+1}\otimes\dots\otimes sb_t)$$

becomes a chain map [6, pp. 21–22]. It follows that the graded dual $(C, d_C) = ((Ts\bar{B})^*, (\tilde{d})^*)$ is a positive differential graded algebra. When B is finite-dimensional over $\mathbf{k}$, C is a finitely generated free algebra (hence $C(z)$ is rational). Because B is locally finite over a field, dualizing does not alter the size of the (co-)homology groups, i.e.,

$$H^i(C, d_C) \approx H_i(Ts\bar{B}, \tilde{d}).$$

Consequently $P_B(z) = H_C(z)$, as promised. □

Remark. The problem of studying Poincaré series of algebraic systems was initially raised in the 1957 paper of A.I. Kostrikin and I. Shafarevich (Dokl.Acad.Nauk S.S.S.R 1115, 1066–1069) where they looked at $P_{\mathbf{k}\oplus N}(z)$ with N a finite-dimensional associative nilpotent algebra over $\mathbf{k}$. They conjectured that such a Poincaré series would always be a rational function. In [17] C. Löfwall disproved that conjecture by establishing rational interdependence between the set of Poincaré-series of finite-dimensional associative $\mathbf{k}$-algebras $\mathbf{k} \oplus N$ with $N^3 = 0$, and the set of Hilbert series of 1–2 algebras G over $\mathbf{k}$. The latter set was known to contain non-rational series by an example of J.B. Shearer [19].

The lower left connection in Fig. 1 follows from this interdependence since restricting G to the class of 1–2 Hopf algebras corresponds to restricting N to being commutative.

3. The reduction to one–two algebras

In this section we will define a 'contiguous presentation' and prove links (c) and (d) of Fig. 1. This will complete the proof of Theorem 1.

Before getting into the proofs themselves, a few remarks on the context of this question on reducing to one-two algebras may be helpful. In [5, Theorem 6(a)] it is shown that the Hilbert series of any f.p. graded algebra G differs 'very little' from the Hilbert series of an associated degree-one-generated algebra N; the precise relationship is that

$$N(z) = p(z)G(z) + q(z)$$

for a pair of polynomials p and q. Theorem 6(b) of [5] goes on to say that there is a one-two $\mathbf{k}$-algebra B such that

$$N(z) \leqslant B(z) \leqslant r(z)N(z) + s(z)$$

for certain polynomials r and s, where $\leqslant$ denotes coefficient-wise inequality. But there is no guarantee in [5] that $B(z) \sim N(z)$, because there is no single formula which describes $B(z)$ in terms of the series $N(z)$ and invariants such as $\mathrm{Tor}_1^N(\mathbf{k}, \mathbf{k})$ or $\mathrm{Tor}_2^N(\mathbf{k}, \mathbf{k})$. Indeed, the precise series $B(z)$ is sensitive to the specific presentation chosen for N.

The ideas developed here were suggested by this seeming intractability. We will present a variation on the construction in [5, 6(b)] and a class $\mathcal{N}$ of d.o.g. algebras such that, when $N \in \mathcal{N}$, the resulting one–two algebra B has a Hilbert series which is rationally related to that of N. The condition defining $\mathcal{N}$ is actually a constraint on N's presentation, and when the condition is met we will call the presentation 'contiguous'. It then remains only to show that any finitely presented graded algebra's Hilbert series is rationally related to the Hilbert series of an algebra in $\mathcal{N}$, which we accomplish through a variation on the construction in [13].

Definition 2. Let G be a degree-one-generated finitely presented graded algebra over $\mathbf{k}$ and let

$$G = \mathbf{k}\langle x_1, \dots, x_g\rangle / \langle \alpha_1, \dots, \alpha_r\rangle \tag{11}$$

be any minimal presentation for G. Let $F = \mathbf{k}\langle x_1, \dots, x_g\rangle$ be the d.o.g. free algebra and for subsets D and E of F let $DE = \operatorname{span}\{xy \mid x \in D, y \in E\}$. Put $R = \{\alpha_1, \dots, \alpha_r\}$ and let $R_i = \{\alpha_j \mid |\alpha_j| = i\}$, so $R = R_2 \cup R_3 \cup \dots \cup R_d$, where $d = \max\{|\alpha_j| \mid 1 \le j \le r\}$. Let I be the two-sided ideal $\langle \alpha_1, \dots, \alpha_r\rangle = \mathrm{FRF}$ in F and let $P = \operatorname{span}\{x_1, \dots, x_g\} = F_1 \subset F$. Then the presentation (11) is called *contiguous* if and only if, for every integer m in the range $2 \le m \le d-1$, we have the equality

$$\left(\sum_{i=m+1}^{d} R_i F\right) \cap (P^m I) = \sum_{i=m+1}^{d} R_i I. \tag{12}$$

Lemma 2. *Any one–two algebra has a contiguous presentation.*

Proof. Condition (12) is vacuously satisfied since $d = 2$ and there are no integers in the range $2 \le m \le d-1$. □

Remark. Lemma 2 justifies the placement of the rectangle for 'contiguous presentation' directly above the region for 'one–two algebras' in Fig. 1.

Example 1. An example of a presentation which is not contiguous is

$$G = \mathbf{k}\langle u, v, w, x, y\rangle / \langle vx - xw, xy, uvx\rangle.$$

To see that (12) is violated, try $m = 2$:

$$\left(\sum_{i=3}^{3} R_i F\right) \cap P^2 I = (uvxF) \cap (P^2 I) \ni uvxy$$

but $\sum_{i=3}^{3} R_i I = uvxI$ contains no non-zero elements in degree four.

Example 2. An example of a presentation which is contiguous is

$$G = \mathbf{k}\langle u, v, w, x, y\rangle / \langle vx - xw, xy, uxw\rangle.$$

To see that (12) is satisfied we have $d=3$ so we need only check $m=2$:

$$\left(\sum_{i=3}^{3} R_iF\right) \cap P^2I = (uxwF) \cap P^2I = ux(wF \cap I)$$

$$= uxwI = R_3I = \sum_{i=3}^{3} R_iI.$$

An algebra's presentation being contiguous is not an invariant of the algebra. Clearly the algebras presented in Examples 1 and 2 are isomorphic! The fact that this condition is nevertheless useful is implicit in

Theorem 7. *Let G be a degree-one-generated* **k***-algebra and suppose that G has a contiguous presentation with g generators. Let F, R_j, I, P, and the number d be as in Definition 2. Then there is a one–two algebra B whose Hilbert series is given by*

$$B(z) = \left(\frac{1}{1-z}\right)G(z)\left[1 + (g^2+g^3+\ldots+g^{d-1})z - \sum_{j=3}^{d} \#(R_j)(z^2+z^3+\ldots+z^{j-1})\right]. \tag{13}$$

Corollary 5 (Link (d)). *The set of Hilbert series of finitely presented graded algebras which have contiguous presentations is rationally dependent upon the set of Hilbert series of one-two algebras.* □

Proof of Theorem 7. Let $S=\{x_1,\ldots,x_g\}$ be the set of generators for G. For $m\geq 0$ let Y_m denote the collection of all integer m-tuples $\sigma=(\sigma_1,\ldots,\sigma_m)$ in which $1\leq\sigma_i\leq g$. For $\sigma\in Y_m$ let x_σ be shorthand for $x_{\sigma_1}x_{\sigma_2}\ldots x_{\sigma_m}\in F$. Put $Y=\bigcup_{m=2}^{d-1} Y_m$ and for each $\sigma\in Y$ let u_σ denote a new symbol of degree one. Define a graded **k**-algebra A by

$$A=\mathbf{k}\langle S\cup\{u_\sigma\}_{\sigma\in Y}\cup\{v\}\rangle/J,$$

where $|v|=1$ and J is the two-sided ideal generated by

$$\{u_\sigma u_\tau \mid \sigma,\tau\in Y\}\cup\{x_iu_\sigma \mid x_i\in S,\ \sigma\in Y\}$$
$$\cup\{vx_j \mid x_j\in S\}\cup\{vu_{(\sigma_1,\sigma_2)}-x_{\sigma_1}x_{\sigma_2} \mid (\sigma_1,\sigma_2)\in Y_2\}$$
$$\cup\{vu_{(\sigma_1\cdots\sigma_m)}-u_{(\sigma_1\cdots\sigma_{m-1})}x_{\sigma_m} \mid (\sigma_1,\ldots,\sigma_m)\in Y_m \text{ and } 3\leq m<d\}.$$

By using the relations in J, any element of A can be written uniquely as a linear combination of elements having the following 'standard form':

$$w=x_\tau v^j \quad\text{or}\quad w=u_\sigma x_\tau v^j; \tag{14}$$

so as graded vector spaces we have $A\approx(\mathbf{k}\oplus U)\otimes F\otimes\mathbf{k}[\upsilon]$, where $U=\mathrm{span}\{u_\sigma\mid\sigma\in Y\}$.

One key property of A is that it is a one–two algebra. The other is that $F=\mathbf{k}\langle S\rangle$ embeds in A in such a way that the equation

$$x_\sigma=\upsilon^{m-1}u_\sigma$$

holds whenever $\sigma=(\sigma_1,\dots,\sigma_m)$ with $2\le m<d$. Consequently the elements of $(F)_{\ge 2}$ all have a common left factor in A. Because of this we can take an arbitrary $y\in F_m$ for $m\le d$, write it as $y=\sum_{|\sigma|=m}c_\sigma x_\sigma$ where $c_\sigma\in\mathbf{k}$, and then define

$$y^{[t]}=\sum_{|\sigma|=m}c_\sigma u_{(\sigma_1\dots\sigma_t)}x_{\sigma_{t+1}}\cdots x_{\sigma_m}$$

for t in the range $2\le t<m$ to get the relation that

$$y=\upsilon^{t-1}y^{[t]} \tag{15}$$

whenever $2\le t<|y|\le d$. Defining $y^{[1]}=y$, note that when any $y^{[t]}$ is written in 'standard form' (14), we have $y^{[t]}\in U^{[t]}\otimes F\otimes 1$, where $U^{[1]}$ is defined to be $\mathbf{k}$ and $U^{[t]}=\mathrm{span}\{u_\sigma\mid\sigma\in Y_t\}$ for $t\ge 2$. Our plan is to take as B a quotient of A in which G instead of F embeds.

For each of the defining relations α_j in the presentation (11) for G, let $\beta_j=\alpha_j^{[|\alpha_j|-1]}$. Let $K=\langle\beta_j\mid\alpha_j\in R\rangle$ be the two-sided ideal they generate in A and put $B=A/K$. Note by (15) that $|y^{[t]}|=|y|-t+1$, so each $|\beta_j|=|\alpha_j|-(|\alpha_j|-1)+1=2$ and B is therefore a one-two algebra. We assert that, in A,

$$K=(\mathbf{k}\otimes I\otimes\mathbf{k}[\upsilon])\oplus\bigoplus_{m=2}^{d-1}(U^{[m]}\otimes I\otimes\mathbf{k}[\upsilon]$$

$$+\mathrm{span}\{\alpha_j^{[m]}F\otimes\mathbf{k}[\upsilon]\mid|\alpha_j|>m\}). \tag{16}$$

To prove (16), the reader may check that the right-hand side is a two-sided ideal which contains each β_j and that a spanning set for the right-hand side is obtained by multiplying the β_j on the right and/or the left by sequences of generators of A.

We are almost done. It remains only to compute $B(z)$. Fortunately each vector space summand involved in (16) is homogeneous in $A=\bigoplus_{n=0}^\infty A_n$, where we describe a $\mathbf{k}$-submodule M of A as *homogeneous* if $M=\bigoplus_{n=0}^\infty(M\cap A_n)$. Because of this we can write $B(z)=A(z)-K(z)$, where

$$K(z)=I(z)\left(\frac{1}{1-z}\right)+\sum_{m=2}^{d-1}V^{[m]}(z)\left(\frac{1}{1-z}\right). \tag{17}$$

Here $V^{[m]}=U^{[m]}\otimes I+\mathrm{span}\{\alpha_j^{[m]}F\mid|\alpha_j|>m\}\subset U^{[m]}\otimes F$ for $m\ge 2$.

To compute $V^{[m]}(z)$, let $\mu_m\colon U^{[m]}\otimes F\to F$ be the degree-$(m-1)$ homomorphism of right F-modules defined by $\mu_m(u_\sigma)=x_\sigma$ for $\sigma\in Y_m$. Then each μ_m is one-to-one and

$$\mu_m(V^{[m]})=P^mI+\sum_{|\alpha_j|>m}\alpha_jF=P^mI+\sum_{i=m+1}^{d}R_iF.$$

Consequently

$$V^{[m]}(z) = z^{-m+1}(P^m I + \sum_{i=m+1}^{d} R_i F)(z)$$

$$= z^{-m+1}\left[P^m I(z) + \left(\sum_{i=m+1}^{d} R_i F\right)(z) - \left(P^m I \cap \sum_{i=m+1}^{d} R_i F\right)(z)\right].$$

Because the presentation $\mathbf{k}\langle S\rangle/\langle R\rangle$ is a contiguous and a minimal presentation, and because F is free, we get

$$V^{[m]}(z) = z^{-m+1}\left[g^m z^m I(z) + \sum_{i=m+1}^{d} \#(R_i) z^i F(z) - \sum_{i=m+1}^{d} \#(R_i) z^i I(z)\right]$$

$$= z g^m I(z) + G(z) z^{-m+1}\left(\sum_{i=m+1}^{d} \#(R_i) z^i\right).$$

Combining this with formula (17) gives

$$B(z) = A(z) - K(z) = (1 + U(z))(F(z))\left(\frac{1}{1-z}\right) - K(z)$$

$$= \left(\frac{1}{1-z}\right)\left[G(z) + U(z)F(z) - \sum_{m=2}^{d-1}(zg^m)I(z)\right.$$

$$\left. - G(z)\sum_{m=2}^{d-1}\sum_{i=m+1}^{d} \#(R_i) z^{i-m+1}\right]$$

$$= \left(\frac{1}{1-z}\right)G(z)\left[1 + z\left(\sum_{m=2}^{d-1} g^m\right) - \sum_{m=2}^{d-1}\sum_{i=m+1}^{d} \#(R_i) z^{i-m+1}\right],$$

which agrees with (13) after one reverses the order of summation. □

In the beginning of this section we stated that the class $\mathcal{N}$ of algebras having contiguous presentations would provide a stepping-stone between general d.o.g. algebras and one–two algebras. Theorem 7 bridged the gap from $\mathcal{N}$ to one–two algebras, so we need only show how to concoct a contiguous from an arbitrary presentation. The inspiration for this construction came from [13], where Jacobsson devised a way to relate an arbitrary finitely presented graded algebra to a Hopf algebra.

Theorem 8. *Let G be any degree-one-generated finitely presented graded* **k***-algebra. Then there is a graded* **k***-algebra C having a contiguous presentation whose Hilbert series is given by*

$$C(z) = (1 - gz)^{-2}(2 - G(z))^{-1}, \quad \textit{where } g = \dim(\mathrm{Tor}_1^G(\mathbf{k}, \mathbf{k})). \qquad (18)$$

Corollary 6 (Link (c)). *The set of Hilbert series of d.o.g. f.p. graded algebras is rationally dependent upon the set of Hilbert series of algebras which have a contiguous presentation.* □

Proof of Theorem 8. Given G, let

$$G = \mathbf{k}\langle x_1, \dots, x_g\rangle / \langle \alpha_1, \dots, \alpha_r\rangle$$

be a minimal presentation for G, where each $|x_i| = 1$ and each $|\alpha_j| \geq 2$. Put $S = \{x_1, \dots, x_g\}$ and let S' and S'' be copies of S, i.e., they are disjoint from but isomorphic to S as graded sets. From them we may build $\mathbf{k}\langle S'\rangle$ and $\mathbf{k}\langle S''\rangle$, which are copies of $\mathbf{k}\langle S\rangle$, and we denote by b' and b'' respectively the elements in $\mathbf{k}\langle S'\rangle$ and $\mathbf{k}\langle S''\rangle$ corresponding to $b \in \mathbf{k}\langle S\rangle$. Let G_+ denote $\bigoplus_{n=1}^{\infty} G_n$ and retain from Section 2 the notation TG_+ for the graded tensor algebra on G_+.

Define the algebra C to be

$$C \approx \mathbf{k}\langle S'\rangle \otimes TG_+ \otimes \mathbf{k}\langle S''\rangle \tag{19}$$

as a graded vector space but define multiplication in C by the rule

$$(a' \otimes (u_1)(u_2)\dots(u_p) \otimes b'') \cdot (c' \otimes (v_1)(v_2)\dots(v_q) \otimes d'')$$
$$= (a' \otimes (u_1)(u_2)\dots(u_p c)(b v_1)(v_2)\dots(u_q) \otimes d'').$$

If $p = 0$ we intend this to be interpreted as

$$(a' \otimes 1 \otimes b'') \cdot (c' \otimes (v_1)\dots(v_q) \otimes d'') = a'c' \otimes (bv_1)\dots(v_q) \otimes d'',$$

and likewise if $q = 0$. The reader can check that this multiplication is associative.

We claim that a presentation for C is

$$C = \mathbf{k}\langle S \cup S' \cup S''\rangle / \langle x_i x_j' - x_i'' x_j,\ x_i' x_j'' - x_j'' x_i',\ \phi(\alpha_t) \mid 1 \leq i, j \leq g; 1 \leq t \leq r\rangle, \tag{20}$$

where $\phi : \mathbf{k}\langle S\rangle_+ \to \mathbf{k}\langle S \cup S' \cup S''\rangle$ is defined by

$$\phi(x_{i_1} \dots x_{i_n}) = x_{i_1} x_{i_2}' \dots x_{i_n}'.$$

To prove this claim, let D be the right-hand side of (20). For an element $u \in \mathbf{k}\langle S\rangle_+$, (u) denotes its image in TG_+ or $T(\mathbf{k}\langle S\rangle)_+$. Consider the surjective algebra homomorphism

$$\mathbf{k}\langle S \cup S' \cup S''\rangle \to C \approx \mathbf{k}\langle S'\rangle \otimes TG_+ \otimes \mathbf{k}\langle S''\rangle$$

given by $s \mapsto 1 \otimes (s) \otimes 1$, $s' \mapsto s' \otimes 1 \otimes 1$, $s'' \mapsto 1 \otimes 1 \otimes s''$. This homomorphism vanishes on the relations defining D and therefore induces a surjective algebra homomorphism

$$\xi : D \to C$$

We will show that ξ is injective (hence an isomorphism) by constructing a left inverse map ξ'.

Consider the $\mathbf{k}$-linear map $T(\mathbf{k}\langle S\rangle_+)\to D$ defined by $(u)\mapsto\overline{\phi(u)}$. This induces a well-defined map $TG_+\to D$, since if $u\in\langle\alpha_1,\ldots,\alpha_r\rangle$ we may write $u=\sum(a_j\alpha_j b_j)$, so $\phi(u)=\sum\phi(a_j\alpha_j b_j)=\sum(a_j''\phi(\alpha_j)b_j')$, yielding $\overline{\phi(u)}=0$. Now we define the $\mathbf{k}$-linear map

$$\xi':\mathbf{k}\langle S'\rangle\otimes TG_+\otimes\mathbf{k}\langle S''\rangle\approx C\to D$$

by $\xi'(a'\otimes(u_1)\ldots(u_p)\otimes b'')=\overline{a'\phi(u_1)\ldots\phi(u_p)b''}$, $\xi'(a'\otimes 1\otimes b'')=\overline{a'b''}$.

As the reader may easily check, ξ' is an algebra homomorphism and $\xi'\xi$ is the identity on the generators from $S\cup S'\cup S''$. Hence $\xi'\xi=\mathrm{id}_D$, which proves the claim (20).

Thus C is also a d.o.g. algebra and (18) follows from (19), so why is the presentation (20) contiguous? Let F, P, I, d, and R_i be as in Definition 2 applied to the presentation (20), so $R_i=\{\phi(\alpha_t)\mid |\alpha_t|=i\}$ for $i\geq 3$.

Let $M\cup\{1\}\subset\mathbf{k}\langle S\rangle$ be an order ideal of monomials such that the image of M in G forms a $\mathbf{k}$-basis for G_+ (see, e.g., Lemma 1.1 of [3] for this). Having chosen M, let

$$N=\{x_\sigma'\phi(w_1)\phi(w_2)\ldots\phi(w_q)x_\tau''\mid q\geq 0,\ w_j\in M\},$$

which is a set of monomials in F whose images form a $\mathbf{k}$-basis for C. Note in particular that

$$I+\mathrm{span}(N)=F\quad\text{while}\quad I\cap\mathrm{span}(N)=0.$$

Because of the form of the relations in R_i we have for $i\geq 3$ that $R_i\subset\{\phi(\alpha_j)\}\subset\mathrm{span}\{\text{all } x_s x_\sigma'\}$ and consequently that $R_iN\subset PN\subset P^mN$ for any $m\geq 2$ and for each $i\geq 3$, so

$$(P^mI)\cap(R_iN)\subset(P^mI)\cap(P^mN)=0.$$

Combining this with the fact that $P^mI\supset R_iI$ for $i>m$, we get

$$\begin{aligned}\left(\sum_{i=m+1}^{d}R_iF\right)\cap(P^mI)&=\left(\bigoplus_{i=m+1}^{d}R_iF\right)\cap(P^mI)\\&=\left(\bigoplus_{i=m+1}^{d}(R_iN\oplus R_iI)\right)\cap(P^mI)\\&=\left(\left(\bigoplus_{i=m+1}^{d}R_iN\right)\oplus\left(\bigoplus_{i=m+1}^{d}R_iI\right)\right)\cap(P^mI)\\&=\bigoplus_{i=m+1}^{d}R_iI=\sum_{i=m+1}^{d}R_iI,\end{aligned}$$

which is precisely the desired condition (12). □

4. Remarks, opinions, and open questions

Having completed the technical part of the paper, we offer in this section some interpretations of its significance. We will comment on the methods used, on the formulas involved, and on the 'classification problem'. We will also touch upon some possible generalizations of rational dependence and mention some open questions.

One intriguing aspect of our work is the essential role played by general (i.e., non-Hopf) non-commutative graded algebras. Most of Sections 2 and 3 dealt in one way or another with these objects. This emphasis is not at all obviously implicit, say, in Theorem 2 or Theorem 3, which talk exclusively about local rings or about loop spaces. Although work on abstract graded algebras has been criticized as being unduly specialized or irrelevant, it now appears that certain topological and/or ring theoretic constructions must be done (or can far more easily be done) in that category.

Each of the links describing a rational dependence $\mathscr{A} \rightarrow \mathscr{B}$ in Fig. 1 is given by an explicit formula, i.e., for each $A(z)$ one has an explicit formula for $B(z)$ which shows that $B(z) \sim A(z)$. Although the formulas are sometimes complicated, they are constructive in the sense that an obvious finite procedure constructs B from A.

Theorem 1, the title theorem asserting that many rational dependencies exist among classes of series, adds a great deal of importance to the 'classification problem'. This problem asks for a simple or concise description of those series which do occur as the loop series of some finite 1-connected CW complex (resp., the Poincaré series of a local Noetherian ring, etc.). That is, given a series $A(z)$, how can one tell whether $A(z)$ equals some $L_X(z)$ (resp. equals some $P_R(z)$, etc.)? The most commonly occurring variation more modestly seeks a description of those series which are rationally related to a loop series (resp. Poincaré series, etc.). In view of Theorem 1 we now know that an answer to the classification problem for, say, loop series would also give an answer for Poincaré series of Noetherian local rings, for Hilbert series of Hopf algebras, and so on. Presumably the fact that one could settle seventeen classification problems at once affords a heightened incentive to work on this question.

Definition 1 is not the only plausible definition for rational dependence, and some generalizations are possible. Starting with the fixed field $\mathbf{k}$ let $\mathbf{k}(z)$ denote the field of all rational functions over $\mathbf{k}$ in the indeterminate z. Obviously $\mathbf{k}(z)$ is a subfield of $\mathbf{k}((z))$, the field of all formal Laurent series with coefficients in $\mathbf{k}$. Given any series $A(z) \in \mathbf{k}((z))$, let $\mathbf{k}(z, A(z))$ denote the smallest subfield of $\mathbf{k}((z))$ which contains both $\mathbf{k}(z)$ and $A(z)$. Thus $A(z)$ is rational if and only if $\mathbf{k}(z, A) = \mathbf{k}(z)$. The formal Laurent series $A(z)$ and $B(z)$ with coefficients in $\mathbf{k}$ are *weakly rationally related* if and only if $\mathbf{k}(z, A(z)) = \mathbf{k}(z, B(z))$.

Obviously $A(z)$ and $B(z)$ are weakly rationally related if they are rationally related, and the converse holds if either A or B is transcendental over $\mathbf{k}(z)$. However, this converse fails when A and B are algebraic. For example, when $\mathbf{k} = \mathbb{Q}$, the two series $A(z)=(1-25z)^{1/5}$ and $B(z)=A(z)^2$ are weakly rationally related but not rationally related. Because $A(z)$ and $B(z)$ have only integral coefficients, one can wonder whether loop series (resp. Poincaré series, etc.) can be rationally related to them. This is but one unsettled instance of the classification problem.

We restricted our attention in Theorem 1 to d.g.a.'s (A, δ) in which $A(z)$ is rational. A more natural treatment of d.g.a.'s permits $A(z)$ to be irrational and considers the pair of series $A(z)$ and $H_A(z)$. We will say that the *(pairs of) series for the d.g.a.'s* (A, δ) and (C, d) are *rationally related* if and only if the field extensions $\mathbf{k}(z, A(z), H_A(z))$ and $\mathbf{k}(z, C(z), H_C(z))$ coincide. Since such a field extension can easily have transcendence degree two over $\mathbf{k}(z)$, we prefer not to compare it with $\mathbf{k}(z, G(z))$ directly but rather to compare it with the extension $\mathbf{k}(z, G(z), N(z))$, where G and N are two possibly unrelated graded $\mathbf{k}$-algebras. If $\mathbf{k}(z, A(z), H_A(z))$ and $\mathbf{k}(z, G(z), N(z))$ agree, we can say that the pair of series of the d.g.a. (A, δ) and the *Hilbert series of the pair* (G, N) are *rationally related.* Note by Theorem 4(D) that the series of a positive d.g.a. (C, d) is necessarily rationally related to the Hilbert series of the pair $(\tilde{C}, C)$. Likewise the [pair of] series of a negative d.g.a. (A, δ) is rationally related to the [pair of] series of a positive d.g.a. (C, d) through Theorem 5.

Lemma 3. *Given any pair (G, N) of finitely presented graded algebras, there is a finitely presented negative d.g.a. (A, δ) whose pair of series is rationally related to $(G(z), N(z))$.*

Proof. Let (B, d) be chosen (see [2] or [15]) such that B is a finitely presented free $\mathbf{k}$-algebra and $H_B(z) \sim G(z)$. Put

$$(A, \delta) = (B, d) \amalg (N, 0) = (B \amalg N, d \amalg 0).$$

The Hilbert and homology series of the coproduct A are related to the respective series of the factor d.g.a.'s via (see [14])

$$A(z)^{-1} = B(z)^{-1} + N(z)^{-1} - 1 \sim N(z)$$

and

$$H_A(z)^{-1} = H_{(B,d)}(z)^{-1} + H_{(N,0)}(z)^{-1} - 1 = H_B(z)^{-1} + N(z)^{-1} - 1.$$

One quickly sees that $\mathbf{k}(z, A(z), H_A(z)) = \mathbf{k}(z, N(z), G(z))$. □

Theorem 9 is the pair-wise version of Theorem 1. To save space not all of the seventeen possible sets are listed.

Theorem 9. *The following sets of pairs of series are all rationally dependent upon each other:*

(a) *The set of pairs $(A(z), H_A(z))$ for a negative d.g.a. (A, δ).*

(b) *The set of pairs $(C(z), H_C(z))$ for a positive d.g.a. (C, d).*

(c) *The set of Hilbert series of pairs of finitely presented graded algebras.*

(d) *The set of Hilbert series of pairs of one-two Hopf algebras with global dimension three.*

(e) *The set of pairs* $(P_R(z), P_S(z))$ *for local Noetherian rings R and S.*

(f) (*When* $\mathbf{k}$ *is a prime field*) *The set of pairs* $(L_W(z), L_X(z))$ *for simply-connected finite CW complexes W and X.*

Proof. That the first three sets are interdependent follows from Lemma 3 and the remarks immediately preceding it. The dependencies of (d), (e), and (f) on (c) and vice versa are obvious consequences of Theorem 1. □

Rational dependence can also be generalized in the direction of series in more than one variable. The *double Poincaré series* of a locally finite connected graded $\mathbf{k}$-algebra G is

$$P'_G(y,z)=\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\dim(\operatorname{Tor}^G_{p,q}(\mathbf{k},\mathbf{k}))y^p z^q.$$

If $A(y,z)$ and $B(y,z)$ are formal Laurent series over $\mathbf{k}$ in two commuting indeterminates, we are tempted to call A and B *rationally related* if $\mathbf{k}(y,z,A(y,z))=\mathbf{k}(y,z,B(y,z))$. One could then ask whether double Poincaré series of general finitely presented graded algebras are rationally dependent upon double Poincaré series of, say, Hopf algebras. We have no results along these lines at this time.

Interestingly, the only positive results about double series dependency which are currently available suggest that a broader notion of rational relationship may be desirable. According to [4], the loop series of a formal space (this is a concept from rational homotopy theory) can be viewed as a double series. When X is a 1-connected formal space with finitely generated rational homology and R is its graded cohomology ring over $\mathbb{Q}$ we have the connection

$$L'_X(yz,z)=P'_R(y,z).$$

Thus we can expect that $\mathbf{k}(y,z,L'_X)\neq\mathbf{k}(y,z,P'_R)$ even though there is a tight relationship between these two series.

References

[1] J.F. Adams and P.J. Hilton, On the chain algebra of a loop space, Comm. Math. Helv. 30 (1955) 305–330.

[2] D. Anick, A counterexample to a conjecture of Serre, Ann. of Math. 115 (1982) 1–33.

[3] D. Anick, Non-commutative graded algebras and their Hilbert Series, J. Algebra 78 (1) (1982) 120–140.

[4] D. Anick, Connections between Yoneda and Pontrjagin algebras, Lecture Notes in Math. 1051 (Springer, Berlin, 1984) 331–350.

[5] D. Anick and C. Löfwall, Hilbert series of finitely presented algebras, in: J.-E. Roos, ed., Algebra, Algebraic Topology, and their Interactions, Lecture Notes in Math. (Springer, Berlin, to appear).
[6] H.J. Baues, Geometry of Loop Spaces and the Cobar Construction, Memoirs A.M.S. 230 (1980).
[7] S. Eilenberg and S. MacLane, On the groups $H(\Pi, n)$, I, Ann. of Math. 58 (1) (1953) 55–106.
[8] S. Eilenberg and J.C. Moore, Homology and Fibrations I, Comm. Math. Helv. 40 (1966) 199–236.
[9] V.K.A.M. Guggenheim and J.P. May, On the Theory and Applications of Differential Torsion Products, memoirs A.M.S. 142 (1974).
[10] T.H. Gulliksen, Homological Invariants of Local Rings, Thesis, Queen's University, Kingston, Canada (1969).
[11] T.H. Gulliksen and G. Levin, Homology of Local Rings, Queen's Papers in Pure and Applied Math. 20 (1969).
[12] S. Halperin and G. Levin, High skeleta of CW complexes, in: J.-E. Roos, ed., Algebra, Algebraic Topology, and their Interactions, Lecture Notes in Math. (Springer, Berlin, to appear).
[13] C. Jacobsson, Finitely presented graded Lie algebras and homomorphisms of local rings, J. Pure Appl. Algebra, in this volume.
[14] J.-M. Lemaire, Algèbres connexes et homologie des espaces de lacets, Lecture Notes in Math. 422 (Springer, Berlin, 1974).
[15] J.-M. Lemaire, Anneaux locaux et espaces de lacets à séries de Poincaré irrationnelles, Lecture Notes in Math. 901 (Springer, Berlin, 1982) 149–156.
[16] G. Levin, Local rings and Golod homomorphisms, J. Algebra 37 (1975) 266–289.
[17] C. Löfwall, Un algèbre dont la série de Poincaré-Betti est non rationnelle, C.R. Acad. Sci. Paris, 288 (1979) Série A, 327–330.
[18] J.-E. Roos. Relations between the Poincaré-Betti series of loop spaces and of local rings, Lecture Notes in Math. 740 (Springer, Berlin, 1979) 285–322.
[19] J.B. Shearer, A graded algebra with non-rational Hilbert series, J. Algebra 62 (1980), 228–231.